Part A

1. C
2. D
3. B
4. B
5. D
6. D
7. D
8. C
9. C
10. A
11. C
12. C
13. A
14. B
15. B
16. B
17. D
18. B
19. C
20. C
21. B
22. C
23. C
24. A
25. B
26. C
27. C
28. D
29. D
30. D

Part B

31. B
32. C
33. A
34. B
35. B
36. C
37. B
38. C
39. B
40. A
41. B
42. C
43. B
44. A
45. D

CALCULUS AB

SECTION I, Part A Solutions

1. Correct answer: (C)

We can rewrite our integral as

$$\int_1^e \frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}\ dx = \int_1^e 1 - \frac{1}{x} + \frac{1}{x^2}\ dx$$

$$= x - \ln x - \frac{1}{x}\Big|_1^e = \left(e - \ln e - \frac{1}{e}\right) - \left(1 - \ln 1 - 1\right) = e - 1 - \frac{1}{e}$$

2. Correct answer: (D)

To find $f'(x)$, we need to use the chain rule.

$$\frac{d}{dx}(\cos(e^{-\tan x})) = -\sin(e^{-\tan x}) \cdot \frac{d}{dx}(e^{-\tan x})$$

$$= -\sin(e^{-\tan x}) \cdot (-\sec^2 x e^{-\tan x})$$

$$= e^{-\tan x}\sin(e^{-\tan x})\sec^2 x$$

3. Correct answer: (B)

We don't need to use the Fundamental Theorem here, we only need to substitute $x = 2$ into the integral.

$$F(2) = \int_0^2 t^3 + 2\ dt = \frac{t^4}{4} + 2t \Big|_0^2 = \left(\frac{16}{4} + 4\right) = 8$$

4. Correct answer: (B)

First, find $f'(x)$.

$$f'(x) = 2 \cdot 2\cos x \cdot \frac{d}{dx}(\cos x) + \sec^2 x$$

$$= 4\cos x(-\sin x) + \sec^2 x$$

$$= -4\sin x \cos x + \sec^2 x$$

Now find $f'\left(\frac{\pi}{6}\right)$.

$$f'\left(\frac{\pi}{6}\right) = -4\sin\frac{\pi}{6}\cos\frac{\pi}{6} + \sec^2\frac{\pi}{6}$$

$$= -4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{2\sqrt{3}}{3}\right)^2$$

$$= -\sqrt{3} + \frac{4}{3}$$

5. Correct answer: (D)

If the function is increasing and concave up, then the first and the second derivatives will be positive.

$$f'(x) = 5x^4 - 30x^2$$

Then we need to find where the function is positive.

$$5x^4 - 30x^2 = 0$$

$$5x^2(x^2 - 6) = 0$$

$$x = -\sqrt{6}, 0, \sqrt{6}$$

We need to evaluate f' at test values $x = -3$, $x = -1$, $x = 1$, and $x = 3$.

$$f'(-3) = 5(-3)^4 - 30(-3)^2 = 405 - 270 = 135 > 0$$

$$f'(-1) = 5(-1)^4 - 30(-1)^2 = 5 - 30 = -25 < 0$$

$$f'(1) = 5(1)^4 - 30(1)^2 = 5 - 30 = -25 < 0$$

$$f'(3) = 5(3)^4 - 30(3)^2 = 405 - 270 = 135 > 0$$

Therefore, the function increases on the interval $(-\infty, -\sqrt{6})$ and $(\sqrt{6}, \infty)$.

$$f''(x) = 20x^3 - 60x$$

$$20x^3 - 60x = 0$$

$$20x(x^2 - 3) = 0$$

$$x = -\sqrt{3}, 0, \sqrt{3}$$

We need to evaluate f'' at the test values of $x = -2$, $x = -1$, $x = 1$, and $x = 2$.

$$f''(-2) = 20(-2)^3 - 60(-2) = -160 + 120 = -40 < 0$$

$$f''(-1) = 20(-1)^3 - 60(-1) = -20 + 60 = 40 > 0$$

$$f''(1) = 20(1)^3 - 60(1) = 20 - 60 = -40 < 0$$

$$f''(2) = 20(2)^3 - 60(2) = 160 - 120 = 40 > 0$$

Therefore, the function is concave up on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$. Therefore, the function is increasing and concave up on the interval $(\sqrt{6},\infty)$, because $\sqrt{3} < \sqrt{6}$.

6. Correct answer: (D)

We should take the derivative of each of I, II, and III, and see what we get.

$$\frac{d}{dx}\left(\frac{\sin^3 x}{3} + \sin^2 x\right) = \frac{3\sin^2 x\cos x}{3} + 2\sin x\cos x = \sin^2 x\cos x + 2\sin x\cos x$$

$$\frac{d}{dx}\left(\frac{2+\sin^3 x}{3} - \cos^2 x\right) = \frac{3\sin^2 x\cos x}{3} - 2\cos x(-\sin x) = \sin^2 x\cos x + 2\cos x\sin x$$

$$\frac{d}{dx}\left(\frac{\cos^3 x}{3} - \frac{\cos^2 x}{2}\right) = \frac{3\cos^2 x(-\sin x)}{3} - \frac{2\cos x(-\sin x)}{2} = -\sin x\cos^2 x + \cos x\sin x$$

Therefore, I and II are both antiderivatives of $f(x)$.

7. Correct answer: (D)

$$\frac{d}{dx}(\cos(e^{3x^2})) = -\sin e^{3x^2} \cdot \frac{d}{dx}e^{3x^2}$$

$$= -\sin e^{3x^2} \cdot e^{3x^2} \cdot \frac{d}{dx}3x^2 = -\sin e^{3x^2} \cdot e^{3x^2} \cdot 6x = -6xe^{3x^2}\sin e^{3x^2}$$

8. Correct answer: (C)

Since speed is the absolute value of velocity, then $|-15| > |10|$ and the maximum speed is reached at $t = 5$ minutes.

9. Correct answer: (C)

If the acceleration is zero, then the slope is zero, or a horizontal line. Therefore, the acceleration is zero on the interval $[2,3]$ and $[6,7]$.

10. Correct answer: (A)

First find the slope, which is the derivative of the curve when $x = \frac{\pi}{2}$.

$$y' = -2\sin 2x + 6\sec^2 2x$$

$$y'\left(\frac{\pi}{2}\right) = -2\sin\pi + 6\sec^2\pi = -2(0) + 6(-1)^2 = 6$$

Using the point-slope form, the equation of the line could be written as

$$y - 3 = 6\left(x - \frac{\pi}{2}\right)$$

$$y - 3 = 6x - 3\pi$$

$$y = 6x - 3\pi + 3$$

$$y - 6x = 3 - 3\pi$$

$$6x - y = 3\pi - 3$$

11. Correct answer: (C)

Since f' is positive on $(-\infty, -2)$, f must be increasing over this same interval. Since f' is negative on $(-2,0)$ and $(2,\infty)$, f must be decreasing over these same intervals.

12. Correct answer: (C)

$$f'(x) = 4x^3 - 30x^2 + 50x = 2x(x-5)(2x-5)$$

Use test values $x = -1$, $x = 1$, $x = 3$, $x = 6$.

$$f'(-1) = 4(-1)^3 - 30(-1)^2 + 50(-1) = -4 - 30 - 50 = -84 < 0$$

$$f'(1) = 4(1)^3 - 30(1)^2 + 50(1) = 4 - 30 + 50 = 24 > 0$$

Therefore, at $x = 0$ the function f has relative minimum because f' changes from negative to positive, and the only false statement is C.

13. Correct answer: (A)

f is decreasing when $f' < 0$. Since the numerator is non-negative, this is only when the denominator is negative.

$$x - 4 < 0$$

$x < 4$, or $(-\infty, 4)$

14. Correct answer: (B)

Use the chain rule

$$h'(x) = g'(f'(x)) \cdot f''(x)$$

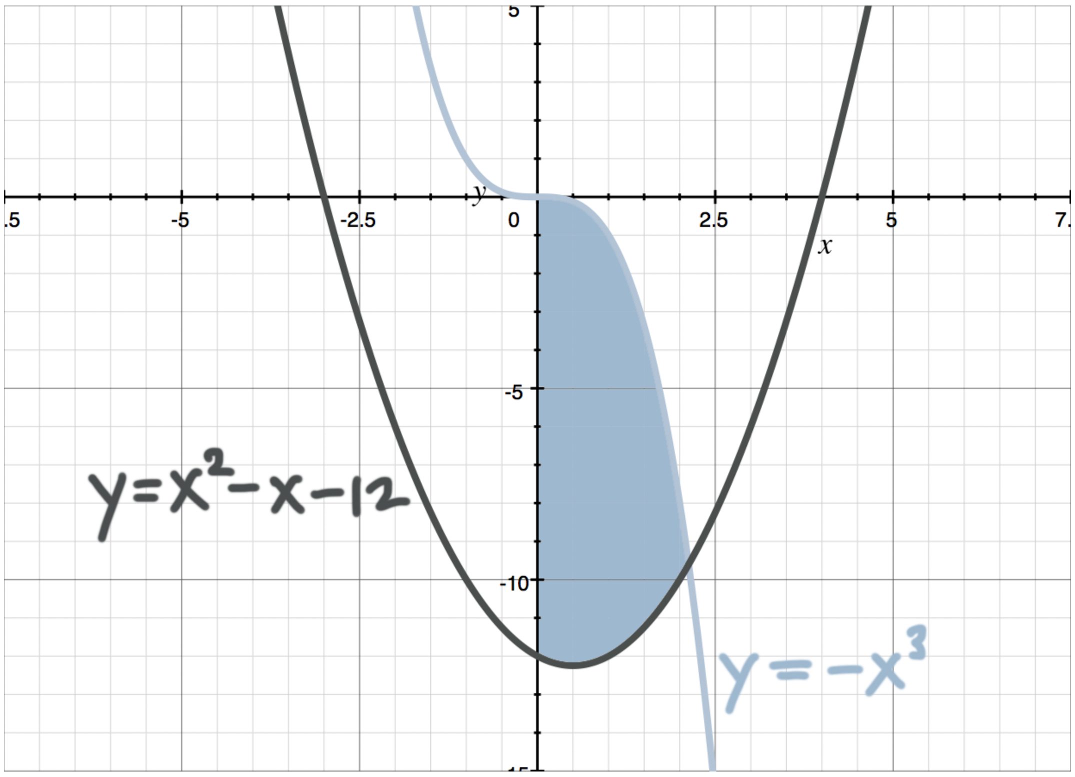
g'' will always be negative. Since $h'(2) = -3$, $f''(2)$ must be positive. Therefore, f must be concave up at $x = 2$.

15. Correct answer: (B)

Since the graph has a vertical tangent at the point $(1,0)$, then the function f is not differentiable at $x = 1$. And the function is discontinuous at $x = 3$, so f is not differentiable at $x = 3$ as well.

16. Correct answer: (B)

Sketch the region.



The curve $y = -x^3$ is above $y = x^2 - x - 12$ over the entire interval. Therefore,

$$A = \int_0^2 -x^3 - (x^2 - x - 12)\ dx = \int_0^2 -x^3 - x^2 + x + 12\ dx$$

$$= -\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + 12x\Big|_0^2 = -4 - \frac{8}{3} + 2 + 24 = \frac{58}{3}$$

17. Correct answer: (D)

Remember that everything is differentiated with respect to x.

$$6x + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 6x$$

$$\frac{dy}{dx}=\frac{3y-6x}{3y^2-3x}$$

$$\frac{dy}{dx}=\frac{y-2x}{y^2-x}$$

18. Correct answer: (B)

Find the antiderivative of each term, and remember to add C at the end to account for the constant.

$$\int x^3+\frac{x}{4}+\frac{2}{x}+5\ dx$$

$$=\frac{x^4}{4}+\frac{x^2}{2\cdot 4}+2\ln x+5x+C=\frac{x^4}{4}+\frac{x^2}{8}+2\ln x+5x+C$$

19. Correct answer: (C)

Find the first and second derivative.

$$f(x)=x^4-6x^3+12x^2$$

$$f'(x)=4x^3-18x^2+24x$$

$$f''(x)=12x^2-36x+24$$

Set the second derivative equal to 0 and solve for x.

$$12x^2-36x+24=0$$

$$12(x^2-3x+2)=0$$

$$x^2 - 3x + 2 = 0$$

$$(x - 1)(x - 2) = 0$$

$$x = 1,\ 2$$

Find the y-values of $f(x)$ when $x = 1$ and $x = 2$.

$$f(1) = 1^4 - 6(1)^3 + 12(1)^2$$

$$f(1) = 1 - 6 + 12$$

$$f(1) = 7$$

and

$$f(2) = 2^4 - 6(2)^3 + 12(2)^2$$

$$f(2) = 16 - 6(8) + 12(4)$$

$$f(2) = 16 - 48 + 48$$

$$f(2) = 16$$

The inflection points are at $(1,7)$ and $(2,16)$.

20. Correct answer: (C)

To find the average value of the function, we should apply the Mean Value Theorem.

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\ dx=\frac{1}{e-1}\int_1^e \frac{3}{x}\ dx=\frac{3}{e-1}\int_1^e \frac{1}{x}\ dx$$

$$=\frac{3}{e-1}(\ln x)\Big|_1^e=\frac{3}{e-1}(\ln e-\ln 1)=\frac{3}{e-1}$$

21. Correct answer: (B)

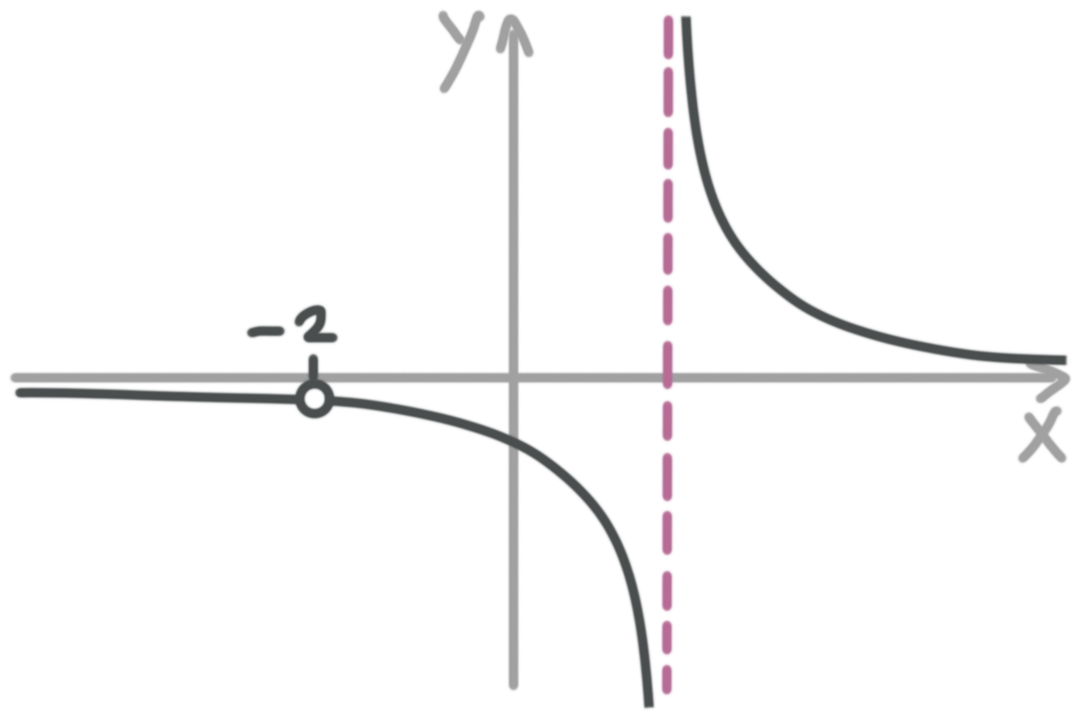
Note that limits at infinity resemble horizontal asymptotes. The limit does not exist when the degree of the numerator is greater than the degree of the denominator, and the limit is zero when the degree of the denominator is greater than the degree of the numerator. When the degrees are equal, the limit is the ratio of the leading coefficients of the numerator and the denominator. Therefore, the limit is zero.

22. Correct answer: (C)

We can rewrite $f(x)$ as

$$f(x)=\frac{2+x}{(x-2)(x+2)}=\frac{1}{x-2}$$

with a point discontinuity at $x=-2$. Sketch a graph.



We can see that the function is concave up to the right of the asymptote, $(2,\infty)$.

23. Correct answer: (C)

$y=4\sqrt{x}$ and $y=4x^2$ have two intersection points, $(0,0)$ and $(1,4)$.

Since $\sqrt{x}>x^2$, then

$$A=\int_0^1 4\sqrt{x}-4x^2\ dx$$

24. Correct answer: (A)

This is just the derivative of $f(x)=3\cos 2x$ evaluated at $x=\frac{\pi}{4}$, $f'(x)=-6\sin 2x$.

$$f'\left(\frac{\pi}{4}\right)=-6\sin\left(2\cdot\frac{\pi}{4}\right)=-6\sin\frac{\pi}{2}=-6$$

25. Correct answer: (B)

Let $u = \sin x$ and $du = \cos x$. So we can rewrite the integral as

$$\int \sin^3 x \cos x \ dx = \int u^3 \ du = \frac{u^4}{4}$$

$$\frac{\sin^4 x}{4}\Bigg|_0^{\frac{\pi}{2}} = \frac{\left(\sin \frac{\pi}{2}\right)^4}{4} - \frac{(\sin 0)^4}{4} = \frac{1}{4} - 0 = \frac{1}{4}$$

26. Correct answer: (C)

The problem asks for the average velocity and we are given the position equation, so we should determine the slope of the secant line.

$$s\left(\frac{3\pi}{2}\right) = 2\sin\frac{3\pi}{2} + \frac{2 \cdot \frac{3\pi}{2}}{\pi} + 4$$

$$= -2 + 3 + 4 = 5$$

$$s(0) = 2\sin 0 + \frac{2 \cdot 0}{\pi} + 4 = 4$$

Then average velocity is

$$v_{avg} = \frac{s\left(\frac{3\pi}{2}\right) - s(0)}{\frac{3\pi}{2} - 0} = \frac{5-4}{\frac{3\pi}{2}} = \frac{1}{\frac{3\pi}{2}} = \frac{2}{3\pi}$$

27. Correct answer: (C)

The one-sided limits are

$$\lim_{x\to 2^-} f(x) = 4\ln 4$$

$$\lim_{x\to 2^+} f(x) = 4\ln 4$$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, then $\lim_{x\to 2} f(x) = 4\ln 4$.

28. Correct answer: (D)

$\frac{x}{15} = \cos\theta$ so $x = 15\cos\theta$.

Differentiating with respect to t gives

$$\frac{dx}{dt} = -15\sin\theta\ \frac{d\theta}{dt}$$

When $x = 9$ units, use the Pythagorean theorem to find the third side and then $\sin\theta$.

$$9^2 + h^2 = 15^2$$

$$h^2 = 225 - 81 = 144$$

$$h = \pm 12$$

Since the side length cannot be negative, then $h = 12$.

$$\sin\theta = \frac{h}{15} = \frac{12}{15}$$

$$\sin\theta = \frac{4}{5}$$

Since $\frac{d\theta}{dt} = -5$, we get

$$\frac{dx}{dt} = -15\left(\frac{4}{5}\right)(-5) = 12 \cdot 5 = 60$$

x increases at a rate of 60 radians per minute.

29. Correct answer: (D)

We need to use quotient and chain rules to evaluate the derivative.

$$f'(x) = \frac{(-3x^2 \sin x^3)e^{2x^2} - 4xe^{2x^2} \cdot \cos x^3}{(e^{2x^2})^2}$$

$$= \frac{e^{2x^2}(-3x^2 \sin x^3 - 4x \cos x^3)}{(e^{2x^2})^2}$$

$$= \frac{-3x^2 \sin x^3 - 4x \cos x^3}{e^{2x^2}}$$

30. Correct answer: (D)

The function is increasing from $x = 0$ to $x = 1$ by $\frac{3}{2}$ squared units, which was determined by finding the area of the triangle. Then the function is increasing from $x = 1$ to $x = 3$ by 2 square units. From $x = 3$ to $x = 6$, it is increasing by $2 + 1 = 3$ square units. From $x = 6$ to $x = 8$ the function is decreasing by 2 square units. We can see that the function's value is greatest at $x = 6$ followed by at $x = 8$, then $x = 3$ and $x = 1$.

Therefore, $f(1) < f(3) < f(8) < f(6)$.

CALCULUS AB

SECTION I, Part B Solutions

31. Correct answer: (B)

Since $f'(x)=5$, the slope of the tangent line is 5. To find the point on the line, set the derivative of f equal to 5 and solve for x. Then, substitute the resulting x-value into f to determine the corresponding value of y.

$$f'(x)=\frac{3}{3x+2}$$

$$\frac{3}{3x+2}=5$$

$$3=15x+10$$

$$-7=15x$$

$$x=-\frac{7}{15}\approx-0.467$$

$$y=\ln\left(3\cdot\left(-\frac{7}{15}\right)+2\right)=\ln\frac{3}{5}\approx-0.512$$

Point-slope form of this equation is

$$y+0.512=5(x+0.467)$$

$$y=5x+1.823$$

32. Correct answer: (C)

From the graph, we see that the function f is not continuous at $x = a$.

33. Correct answer: (A)

The graph of $y = x^2 + 2x - 8$ is a parabola that changes from positive to negative at $x = -4$ and from negative to positive at $x = 2$. Since g is always positive, f' does not change sign.

Therefore, f has a relative maximum at $x = -4$ and a relative minimum at $x = 2$.

34. Correct answer: (B)

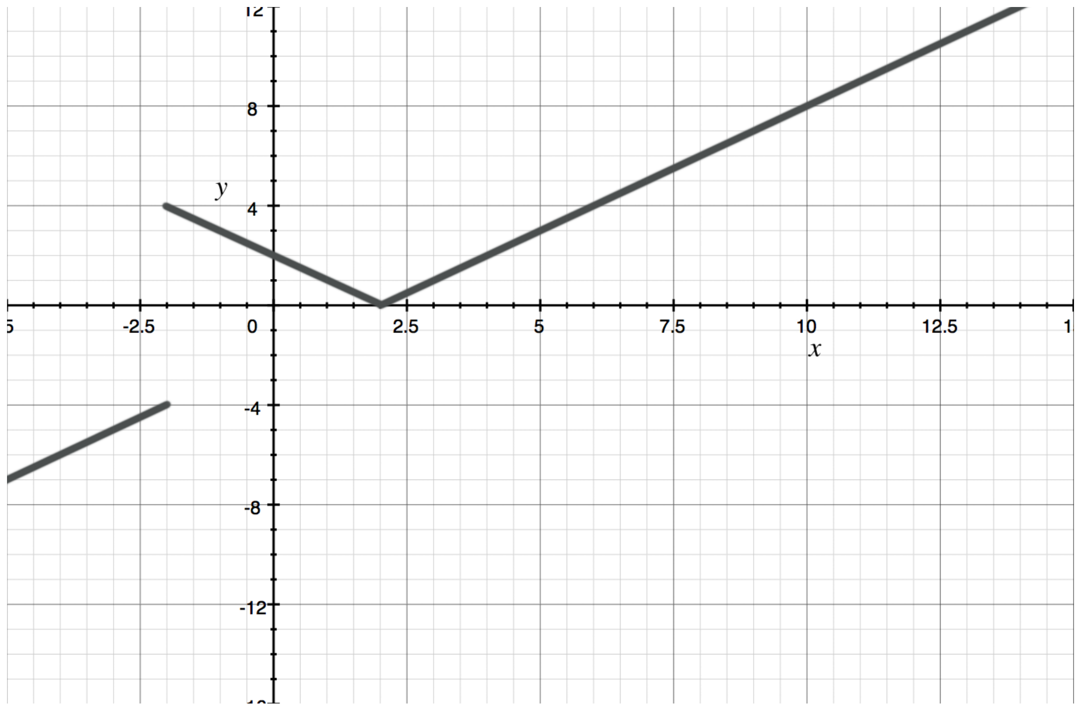
The area of a circle is $A = \pi r^2$. Differentiating with respect to t gives us

$$\frac{dA}{dt} = 2\pi r \ \frac{dr}{dt}$$

We know that $C = 2\pi r$ and $\frac{dr}{dt} = 0.15$, so $\frac{dA}{dt} = 0.15C$.

35. Correct answer: (B)

Graph the function.



We see that it is continuous at $x=2$ and has a jump discontinuity at $x=-2$. Therefore, it is not differentiable at $x=-2$. Since the graph is decreasing before and increasing after $x=2$, there is a local minimum at $x=2$. Therefore, I and II are false.

36. Correct answer: (C)

Since $F(x)$ is an antiderivative of $\frac{(\ln 2x)^4}{x}$, then $F(x)=\int \frac{(\ln 2x)^4}{x}\ dx$.

Let $u=\ln 2x$ and $du=\frac{2}{2x}=\frac{1}{x}\ dx$.

$$F(x)=\int \frac{(\ln 2x)^4}{x}\ dx=\int u^4\ du=\frac{u^5}{5}+c=\frac{(\ln 2x)^5}{5}+c$$

Since $F\left(\frac{1}{2}\right)=2$, then

$$\frac{\left(\ln 2\left(\frac{1}{2}\right)\right)^5}{5}+c=2$$

$$\frac{(\ln 1)^2}{5}+c=2$$

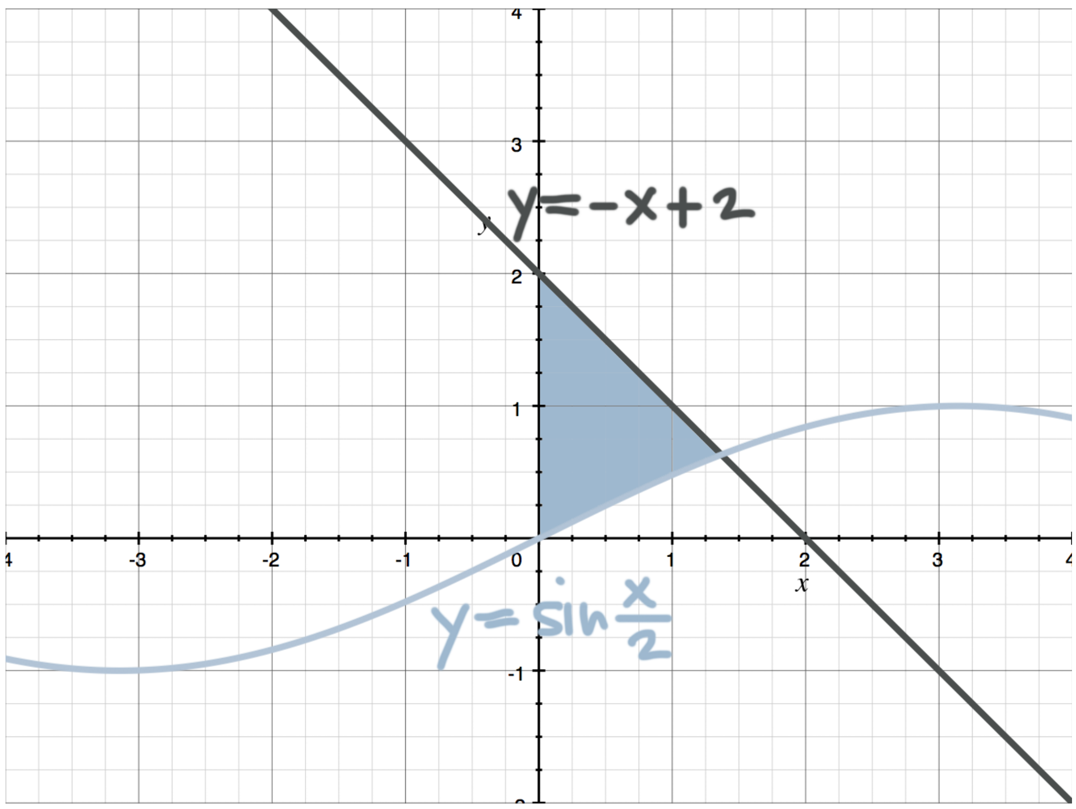
$$c=2$$

Therefore

$$F(x)=\frac{(\ln 2x)^5}{5}+2 \text{ and } F(5)=\frac{(\ln 10)^5}{5}+2=14.945$$

37. Correct answer: (B)

Sketch the functions.



We need to find where these two curves intersect.

$$\sin\frac{x}{2} = -x+2$$

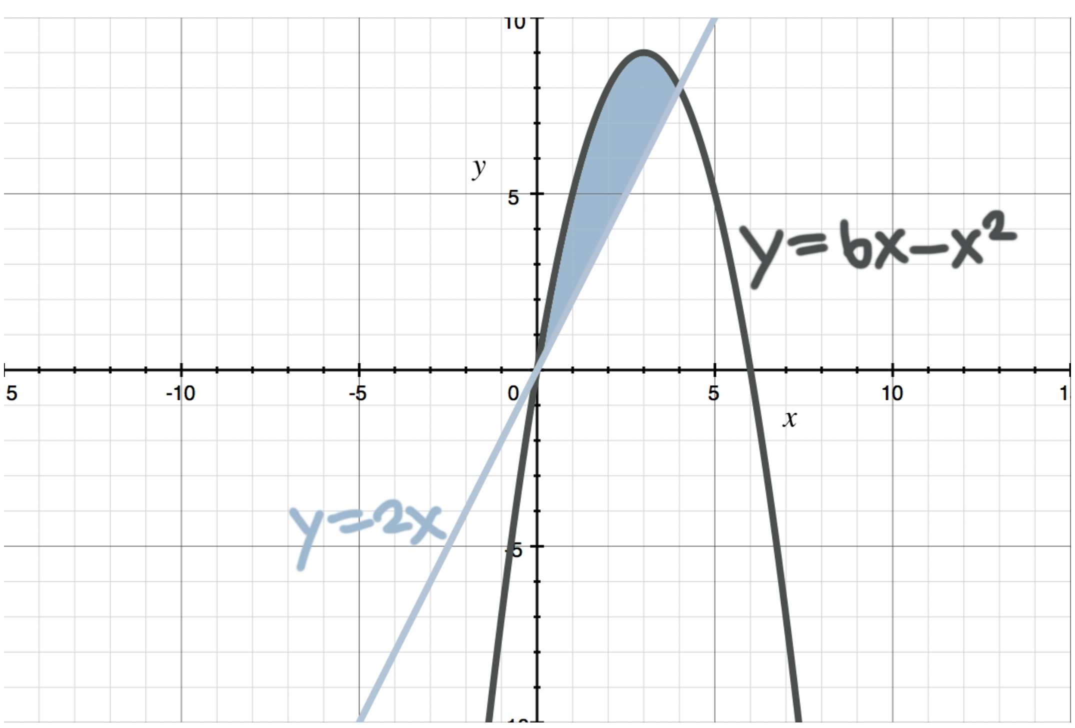
$$x \approx 1.368$$

Since the graph of $y = \sin\frac{x}{2}$ is below the graph of $y = -x+2$, we integrate to find the area.

$$A = \int_0^{1.368} -x+2-\sin\frac{x}{2}\ dx \approx 1.350$$

38. Correct answer: (C)

Sketch the graph.



Now find the points of intersection.

$$2x = 6x - x^2$$

$$0 = 4x - x^2$$

$$x(4 - x) = 0$$

$$x = 0, 4$$

To find the volume about the y-axis, we use

$$V = 2\pi \int_0^4 x(6x - x^2 - 2x)\ dx$$

$$V = 2\pi \cdot \frac{64}{3} \approx 134.041$$

39. Correct answer: (B)

Parallel tangents will be when the slopes of f and g are equal. First, find the derivative of the functions.

$$f'(x) = 2e^{3x} \cdot 3 = 6e^{3x}$$

$$g'(x) = 15x^2 - 2$$

Find x where $f'(x) = g'(x)$.

$$6e^{3x} = 15x^2 - 2$$

$$x \approx -0.478$$

40. Correct answer: (A)

g is an antiderivative of f, so f is the derivative of g. As seen from the graph, the derivative of g changes from negative to positive at $x = a$. Therefore, g has a minimum at $x = a$.

41. Correct answer: (B)

We need to find the second derivative of f and find its x-intercept.

$$f'(x) = 6x^2 + 6x - 2\cos x$$

$$f''(x) = 12x + 6 + 2\sin x$$

$$12x + 6 + 2\sin x = 0$$

$$x \approx -0.430$$

42. Correct answer: (C)

Since $F'(x) = f(x)$, then $F(x) = \int f(x)\ dx$. Therefore, we get

$$F(6) - F(0) = \int_0^6 f(x)\ dx$$

From the graph, we can write

$$F(6) - F(0) = \int_0^2 f(x)\ dx + \int_2^4 f(x)\ dx + \int_4^6 f(x)\ dx$$

where $\int_2^4 f(x)\ dx$ is the area of the rectangle and $\int_4^6 f(x)\ dx$ is the area of the triangle. Therefore

$$F(6) - F(0) = 4.2 + 2(3) + \frac{1}{2}(2)(3) = 4.2 + 6 + 3 = 13.2$$

43. Correct answer: (B)

The right-hand sum is

$$f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_6)\Delta t$$

Since we have six equal subintervals of $[0,12]$, we get

$$2(2.3 + 3.1 + 1.0 + 4.5 + 6.2 + 4.6) = 2(21.7) = 43.4$$

44. Correct answer: (A)

We know that $v(t) = s'(t)$. Since we have linear position, therefore, we get constant velocity, and the correct graph is A.

45. Correct answer: (D)

$$\lim_{x\to a} \frac{x^3 - a^3}{x^2 - a^2} = \lim_{x\to a} \frac{(x-a)(x^2 + ax + a^2)}{(x-a)(x+a)}$$

$$= \lim_{x\to a} \frac{x^2 + ax + a^2}{x + a} = \frac{a^2 + a^2 + a^2}{2a} = \frac{3a^2}{2a} = \frac{3}{2}a$$

CALCULUS AB

SECTION II, Part A Solutions

1. Solution:

a. To find the average value, we use the following formula:

$$r_{avg} = \frac{1}{20-0}\int_0^{20} \frac{1}{10}(h^2-6)\ dh$$

$$= \frac{1}{200}\left(\frac{h^3}{3}-6h\right)\Bigg|_0^{20} = \frac{1}{200}\left(\frac{20^3}{3}-6(20)\right) = \frac{1}{200}\cdot\frac{7{,}640}{3} = \frac{191}{15}\text{ cm}$$

b. $V = \pi\int_0^{20}\left(\frac{1}{10}(h^2-6)\right)^2\ dh$

$$= \frac{\pi}{100}\int_0^{20} h^4-12h^2+36\ dh = \frac{\pi}{100}\left(\frac{h^5}{5}-4h^3+36h\right)\Bigg|_0^{20}$$

$$= \frac{\pi}{100}\left(\frac{20^5}{5}+4(20^3)+36(20)\right) = \frac{\pi}{100}(640{,}000-32{,}000+720)$$

$$= \frac{60{,}832\pi}{10} = \frac{30{,}436\pi}{5}\text{ cm}^3$$

c. $\frac{r}{h} = \frac{10}{20} = \frac{1}{2}$, so $r = \frac{1}{2}h$.

The volume of cone is

$$V = \frac{1}{3}\pi r^2 h$$

So we get

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

Then

$$\frac{dV}{dt} = \frac{1}{12}\pi\left(3h^2\ \frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2\ \frac{dh}{dt}$$

If $h = 5$ and $\frac{dh}{dt} = -\frac{2}{5}$, then

$$\frac{dV}{dt} = \frac{1}{4}\pi(5)^2\left(-\frac{2}{5}\right) = -\frac{5}{2}\pi \text{ cm}^3\text{/hr}$$

2. Solution:

a. We remember that $a(t) = v'(t)$

$$a(t) = \frac{2x - 1}{x^2 - x + 1}$$

$$a(3) = \frac{2(3) - 1}{9 - 3 + 1} = \frac{5}{7}$$

$$v(3) = \ln(3^2 - 3 + 1) = \ln 7 \approx 1.946$$

Since $a(3) > 0$ and $v(3) > 0$, then speed is increasing.

b. The particle changes direction when $v(t) = 0$.

$$\ln(t^2 - t + 1) = 0$$

when

$$t^2 - t + 1 = 1$$

$$t^2 - t = 0$$

$$t(t - 1) = 0$$

$$t = 0,\ 1$$

So

$v(t) < 0$ for $0 < t < 1$

$v(t) > 0$ for $1 < t < 7$

Therefore, the particle changes direction when $t = 1$.

c. To find the position of the particle at time t, we need to use

$$s(t) = s(0) + \int_0^t \ln(u^2 - u + 1)\ du$$

We substitute $s(0) = 5$ to get

$$s(3) = 5 + \int_0^3 \ln(u^2 - u + 1)\ du \approx 5 + 1.915 \approx 6.915$$

d. $\frac{1}{2}\int_2^4 |v(t)|\ dt = \frac{1}{2}\int_2^4 |\ln(t^2 - t + 1)|\ dt = \frac{1}{2} \cdot 3.816 \approx 1.908$

CALCULUS AB

SECTION II, Part B Solutions

3. Solution:

a. $g(-2)=\int_{-3}^{-2} f(t)\ dt=\frac{1}{2}(2)(2)=2$ because the area of a triangle is $\frac{1}{2}bh$.

$$g'(x)=f(x)$$

$$g'(-2)=f(-2)=1$$

$$g''(x)=f'(x)$$

$$g''(-2)=f'(-2)$$

The slope of f is

$$m=\frac{-2+3}{1-2}=\frac{1}{-1}=-1$$

$$g''(-2)=-1$$

b. Since $g'(x)=f(x)$ and $f(x)\le 0$ for $0\le x\le 2$, the function g has neither a relative minimum nor a relative maximum at $x=1$.

c. To find where g has a relative minimum, we need to find where $g'(x)=0$ and where g' changes from negative to positive.

$$g'(x)=f(x)$$

Therefore, $g'(x)=0$ where $f(x)=0$: $x=-1$, $x=1$, and $x=3$.

Since $g' = f$ changes from negative to positive only at $x = 3$, then g has a relative minimum at $x = 3$.

d. The graph of g has a point of inflection at $x = -3$, $x = 0$, $x = 1$, and $x = 2$ because $g'(x) = f(x)$ changes from increasing to decreasing at $x = -3$ and $x = 1$ and from decreasing to increasing at $x = 0$ and $x = 2$.

4. Solution

a. To sketch a slope field, we need to find slope at each given point.

$$\frac{dy}{dx}(1,0) = \frac{-3(1)^3}{0^3}$$

Because this value is undefined, we have the vertical line at $(1,0)$ and the same vertical line for $(-1,0)$.

$$\frac{dy}{dx}(1,1) = \frac{-3(1)^3}{1^3} = -3$$

$$\frac{dy}{dx}(1,2) = \frac{-3(1)^3}{2^3} = -\frac{3}{8}$$

$$\frac{dy}{dx}(0,1) = \frac{-3(0)^3}{1^3} = 0$$

$$\frac{dy}{dx}(0,2) = 0$$

$$\frac{dy}{dx}(0,-1) = 0$$

$$\frac{dy}{dx}(0,-2) = 0$$

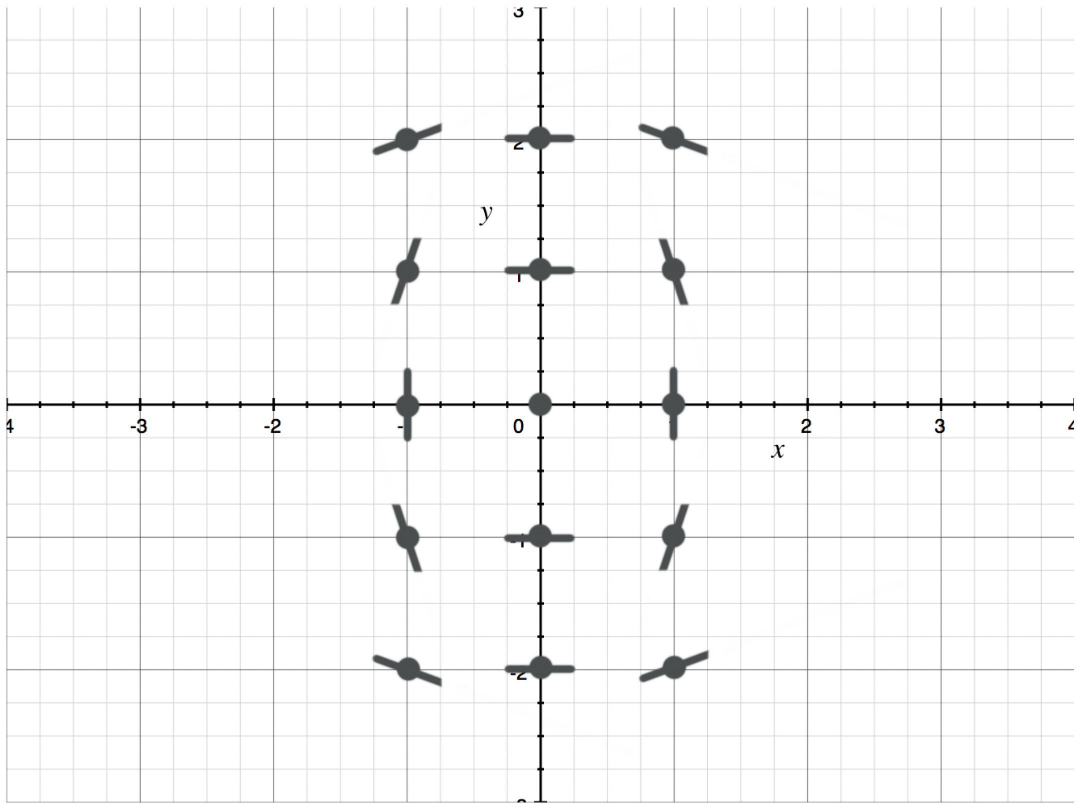
$$\frac{dy}{dx}(-1,1) = \frac{-3(-1)^3}{1^3} = 3$$

$$\frac{dy}{dx}(-1,2) = \frac{-3(-1)^3}{2^3} = \frac{3}{8}$$

$$\frac{dy}{dx}(-1,-1) = \frac{-3(-1)^3}{(-1)^3} = -3$$

$$\frac{dy}{dx}(-1,-2) = \frac{-3(-1)^3}{(-2)^3} = -\frac{3}{8}$$

$$\frac{dy}{dx}(1,-1) = \frac{-3(1)^3}{(-1)^3} = 3 \qquad \frac{dy}{dx}(1,-2) = \frac{-3(1)^3}{(-2)^3} = \frac{3}{8}$$



b. The slope at $(1,2)$ is

$$\frac{dy}{dx}(1,2) = -\frac{3}{8}$$

Therefore, the line tangent to f at $(1,2)$ is

$$y - 2 = -\frac{3}{8}(x-1)$$

$$y = -\frac{3}{8}x + \frac{3}{8} + 2$$

$$y = -\frac{3}{8}x + \frac{19}{8}$$

$$f(1.1) = -\frac{3}{8}(1.1) + \frac{19}{8} = 1.963$$

c. $\frac{dy}{dx} = -\frac{3x^3}{y^3}$

$$y^3\ dy = -3x^3\ dx$$

$$\int y^3\ dy = \int -3x^3\ dx$$

$$\frac{y^4}{4} = -\frac{3}{4}x^4 + C$$

Since $f(1) = 2$, substitute and solve for C.

$$\frac{2^4}{4} = -\frac{3}{4}(1)^4 + C$$

$$4 = -\frac{3}{4} + C$$

$$C = 4 + \frac{3}{4} = \frac{19}{4}$$

Therefore

$$\frac{y^4}{4} = -\frac{3}{4}x^4 + \frac{19}{4}$$

or

$$y^4 = -3x^4 + 19$$

Since the particular solution goes through $(1,2)$, y must be positive, so the particular solution is $y = \sqrt[4]{19 - 3x^4}$.

5. Solution:

a. $f'(6) = \frac{f(7) - f(5)}{7 - 5} = \frac{3 - (-2)}{2} = \frac{5}{2} = 2.5$

b. $\int_1^{12} 2f'(x) - 5\ dx = 2\int_1^{12} f'(x)\ dx - \int_1^{12} 5\ dx$

We know that

$$\int_1^{12} f'(x)\ dx = f(12) - f(1)$$

Therefore, we get

$$\int_1^{12} 2f'(x) - 5\ dx = 2(f(12) - f(1)) - 5x\Big|_1^{12}$$

$$= 2(5 - 2) - 5(12 - 1) = 2 \cdot 3 - 5 \cdot 11 = 6 - 55 = -49$$

c. $\int_1^{12} f(x)\ dx \approx f(1)(2 - 1) + f(2)(5 - 2) + f(5)(7 - 5) + f(7)(8 - 7) + f(8)(12 - 8)$

$$= 2(1) + 0(3) + (-2)(2) + 3(1) + 7(4)$$

$$= 2 - 4 + 3 + 28 = 29$$

d. To find the equation of the tangent line, we use the formula

$$y = mx + b$$

where $(x, y) = (5, -2)$ and $m = 2$. Substitute and solve for b.

$$-2 = 2(5) + b$$

$$-2 - 10 = b$$

$$b = -12$$

Therefore, we get $y = 2x - 12$. Now we need to find $f(6)$.

$$f(6) = 2(6) - 12 = 0$$

Since $f''(x) < 0$ on $5 \leq x \leq 7$, then 0 is the greatest possible value of $f(6)$.

6. Solution:

a. To explain this, we need to evaluate the following

$$\frac{k(4) - k(2)}{4 - 2}$$

$$k(4) = g(f(4)) + 5 = g(4) + 5 = 10 + 5 = 15$$

$$k(2) = g(f(2)) + 5 = g(3) + 5 = 7 + 5 = 12$$

Then

$$\frac{k(4) - k(2)}{4 - 2} = \frac{15 - 12}{2} = \frac{3}{2}$$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $2 < c < 4$, such that $k'(c) = 1.5$.

b. Since $h(x) = \int_1^{f(x)} g(x)\ dx$, then

$$h'(x) = g(f(x)) \cdot f'(x)$$

Therefore

$$h'(2) = g(f(2)) \cdot f'(2)$$

$$= g(3) \cdot 4 = 7 \cdot 4 = 28$$

c. Since $g(2) = 3$, then $g^{-1}(3) = 2$.

$$(g^{-1})'(3) = \frac{1}{g'(g^{-1}(3))} = \frac{1}{g'(2)} = \frac{1}{1} = 1$$

The equation of the tangent line is

$$y - 2 = 1(x - 3)$$

$$y = x - 1$$

d. $\int_2^4 f''\left(\frac{x}{2}\right)\ d = 2f'\left(\frac{x}{2}\right)\Big|_2^4$

$$= 2(f'(2) - f'(1))$$

$$= 2(4 - (-2)) = 12$$